

## Martices 1

- Note: The official CIE book covers a lot of the stuff that Hodder version and probably Collins version didn't cover in this chapter, we'll follow CIE book.

A matrix is something that looks like

$$A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ a_{m1} & \vdots & \vdots & a_{mn} \end{pmatrix}$$

order = row  $\times$  column

## Special Martices

1. zero matrix, denote by  $O_{m \times n}$  where

$$O_{m \times n} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2. Identity matrix which is a square matrix (matrix such that order is  $n \times n$ )

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

the diagonal line all equals to one and the other element all equals to 0.

- 3.

## Addition, Subtraction and Scalar product of a matrix

Consider two random matrices  $A_{m \times n}$  and  $B_{m \times n}$  and the elements inside are denoted by  $a_{ij}, b_{ij}$

$$A + B = (a_{ij} + b_{ij})$$

e.g

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & 9 \end{pmatrix}$$

we can trivially infer that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

$$\mathbf{A} + \mathbf{O} = \mathbf{A}$$

for any matrix  $\mathbf{A}$  there exists a matrix  $\mathbf{N}$  such that

$$\mathbf{A} + \mathbf{N} = \mathbf{O}$$

consider a scalar  $\lambda$  we define that

$$\lambda \mathbf{A} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots \\ \lambda a_{21} & \lambda a_{22} & \dots \\ \dots & \dots & \ddots \end{pmatrix}$$

for subtraction we can infer that

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$$

the rules are same.

## Product of matrices

Consider two matrices  $\mathbf{A}_{m \times p} = (a_{ik})$  and  $\mathbf{B}_{p \times n} = (b_{kj})$

And

$$\mathbf{C}_{m \times n} = \mathbf{A} \times \mathbf{B} = (c_{ij})$$

Here.

$$c_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$$

Notice that only the equal column of the first matrix and the row of the second matrix produces a valid product.

e.g

$$\begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \text{row1} \times \text{col1} & \text{row1} \times \text{col2} \\ \text{row2} \times \text{col1} & \text{row2} \times \text{col2} \end{pmatrix}$$
$$= \begin{pmatrix} -1 \times 0 + 0 \times 1 + 1 \times 3 & -1 \times 3 + 0 \times 2 + 1 \times 1 \\ -1 \times 0 + 1 \times 1 + 3 \times 3 & -1 \times 3 + 1 \times 2 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 10 & 2 \end{pmatrix}$$

**Notice that  $AB \neq BA$**

It also holds that  $AI = IA = A$

**Inverse matrix:** A inverse matrix of  $A$  (hereby  $A$  is a square matrix) is denoted by  $A^{-1}$  such that  $AA^{-1} = I = A^{-1}A$ . A matrix can have no inverse matrix, that kind of matrix is called a **singular matrix**.

The inverse of a matrix is an important stuff that we will talk about later. If you do not understand det notation and singular matrix, its okay.

## System of equations $Ax = b$

Consider  $n$  linear equations with  $n$  variables.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n \end{cases}$$

we denote that

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & a_{33} & \dots \\ a_{m1} & \vdots & \vdots & a_{mn} \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Here notice that  $\mathbf{x}$  and  $\mathbf{b}$  can be vectors.  $n$ -dimesional vectors are  $n \times 1$  martices.

the equations can be rewritten into

$$A\mathbf{x} = \mathbf{b}$$

by multiplying  $A^{-1}$  to both sides it can be easily discovered that

$$\mathbf{x} = \mathbf{A}^{-1} \mathbf{b}$$

**Note:** this equation hides an important idea, can you use this equation to solve for  $\mathbf{A}^{-1}$ ?

Here we can have a good way of judging whether  $\mathbf{A}$  is singular.

*claim.* A square matrix is singular if non-zero vector  $\mathbf{x}$  ( $\mathbf{x} \neq \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$ ) such that  $\mathbf{A} \mathbf{x} = 0$  exists.

e.g: judge whether matrix  $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$  is singular

*solution.* Consider the system of equations

$$\begin{cases} 3x + y = 0 \\ 6x + 2y = 0 \end{cases}$$

we can easily observe that there exists a non-zero solution where  $x = -1, y = 3$ . therefore

$$\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

According to the claim, the matrix is singular.

## Gauss Elimination

Gauss invented a way of solving a system of equations that has  $n$  variables.

e.g use gauss elimination to solve

$$\begin{cases} x + y + 2z = 1 \\ 3x + 4y + 6z = 3 \\ -2x + 3y - 3z = -1 \end{cases}$$

*solution:* first we rewritten the whole equation into form  $\mathbf{A} \mathbf{x} = \mathbf{b}$

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

Then we write LHS into augmented matrix form

$$AM = \left( \begin{array}{cccc|c} 1 & 1 & 2 & \vdots & 1 \\ 3 & 4 & 6 & \vdots & 3 \\ -2 & 3 & -3 & \vdots & -1 \end{array} \right)$$

Also can be written as

$$AM = \left( \begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 3 & 4 & 6 & 3 \\ -2 & 3 & -3 & -1 \end{array} \right)$$

(A-level use the former one)

you can see that we just ignore the  $\mathbf{x}$  vector and write the coefficients together, this is because variables have no significance to the elimination process and this form is better-looking.

now I denote each row as  $R_n$ , the element of  $A_M$  can be represented by  $a_{mn}$ . we now set the pivot as  $a_{11}$  (at the  $n$ -th process, the  $n$ -th pivot is usually  $a_{nn}$ ). The first pivot is usually 1.

**Note:** you can freely move these rows because you're only solving the equation, not finding the inverse of a matrix. However I personally suggest you don't move rows because this will make you confuse.

Now in order to elimination the first column, we consider make  $R_2 \rightarrow R_2 - 3R_1$ , this gives us.

$$\left( \begin{array}{cccc|c} 1 & 1 & 2 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 0 \\ -2 & 3 & -3 & \vdots & -1 \end{array} \right)$$

now  $a_{22}$  is the second pivot. we consider make  $R_3 \rightarrow R_3 + 2R_1$

$$\left( \begin{array}{cccc|c} 1 & 1 & 2 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 5 & 1 & \vdots & 1 \end{array} \right)$$

Now we consider  $R_3 \rightarrow R_3 - 5R_2$

$$\left( \begin{array}{cccc|c} 1 & 1 & 2 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 1 \end{array} \right)$$

Here you can end the process because we have achieved the *echelon form* on the left side.

An *echelon form* is a kind of a matrix such that

$$\begin{pmatrix} \square & * & * & * & \dots \\ 0 & \square & * & * & \dots \\ 0 & 0 & \square & * & \dots \\ 0 & 0 & 0 & \square & \dots \end{pmatrix}$$

here  $\square$  denotes non-zero element,  $*$  can be any element.

**if echelon form cannot be achieved during the elimination process, then  $A$  is singular**

now rewrite the system of equations after gauss elimination.

$$\begin{cases} x + y + 2z = 1 \\ -y = 0 \\ z = 1 \end{cases}$$

Now we can easily solve the equation

$$\begin{cases} x = -1 \\ y = 0 \\ z = 1 \end{cases}$$

The determinant  $\det(\mathbf{A}_{n \times n})$  can be calculated with  $\mathbf{A}$ 's echelon form  $\mathbf{E} = (e_{ij})$

$$\det(\mathbf{A}) = \prod_{k=1}^n e_{kk} \times (-1)^{\frac{n(n-1)}{2}}$$

## Gauss-Jordan's idea

Gauss-Jordan's idea is useful for finding a matrix's inverse. the core idea is to create an augmented matrix form such that

$$\left( \begin{array}{ccc} & \vdots & \\ \mathbf{A} & \vdots & \mathbf{I} \\ & \vdots & \end{array} \right)$$

After gauss's elimination, the result should look like.

$$\left( \begin{array}{ccc} & \vdots & \\ \mathbf{I} & \vdots & \mathbf{A}^{-1} \\ & \vdots & \end{array} \right)$$

e.g Find the inverse of matrix :

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 8 \\ 5 & 1 & 9 \end{pmatrix}$$

*solution:*

Start with the augmented matrix  $[A : I]$ :

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 7 & 8 & 0 & 1 & 0 \\ 5 & 1 & 9 & 0 & 0 & 1 \end{array} \right)$$

$$R_2 \rightarrow R_2 - 4R_1, R_3 \rightarrow R_3 - 5R_1$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -4 & -4 & 1 & 0 \\ 0 & -9 & -6 & -5 & 0 & 1 \end{array} \right)$$

$$R_2 \rightarrow -R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 4 & -1 & 0 \\ 0 & -9 & -6 & -5 & 0 & 1 \end{array} \right)$$

$$R_3 \rightarrow R_3 + 9R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 4 & -1 & 0 \\ 0 & 0 & 30 & 31 & -9 & 1 \end{array} \right)$$

$$R_3 \rightarrow \frac{1}{30}R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 4 & -1 & 0 \\ 0 & 0 & 1 & \frac{31}{30} & -\frac{3}{10} & \frac{1}{30} \end{array} \right)$$

$$R_1 \rightarrow R_1 - 3R_3, R_2 \rightarrow R_2 - 4R_3$$

$$\left( \begin{array}{ccc|ccc} 1 & 2 & 0 & : & -\frac{7}{10} & \frac{9}{10} & -\frac{1}{10} \\ 0 & 1 & 0 & : & -\frac{7}{15} & \frac{1}{5} & -\frac{2}{15} \\ 0 & 0 & 1 & : & \frac{31}{30} & -\frac{3}{10} & \frac{1}{30} \end{array} \right)$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & 0 & : & -\frac{7}{30} & \frac{1}{2} & \frac{1}{30} \\ 0 & 1 & 0 & : & -\frac{7}{15} & \frac{1}{5} & -\frac{2}{15} \\ 0 & 0 & 1 & : & \frac{31}{30} & -\frac{3}{10} & \frac{1}{30} \end{array} \right)$$

Therefore:

$$\mathbf{A}^{-1} = \left( \begin{array}{ccc} -\frac{7}{30} & \frac{1}{2} & \frac{1}{30} \\ -\frac{7}{15} & \frac{1}{5} & -\frac{2}{15} \\ \frac{31}{30} & -\frac{3}{10} & \frac{1}{30} \end{array} \right)$$

## Determinants

determinant of a  $2 \times 2$  matrix

$$\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

determinant of a  $3 \times 3$  matrix

$$\det(\mathbf{A}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + dhc - gec - bdi - hfa$$

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det(\mathbf{A}) \times \det(\mathbf{B})$$

## Matrix transformations

In general, a two-way stretch of scale factor  $a$  in the  $x$ -direction and scale factor  $b$  in the  $y$ -direction is represented by the matrix  $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ .

In general, a shear parallel to the  $x$ -axis, with a shear factor of  $k$  and parallel to the  $y$ -axis, with a shear factor of  $l$ , is represented by the matrix  $\begin{pmatrix} 1 & k \\ l & 1 \end{pmatrix}$ .

In general, a reflection in the line  $y = x \tan \theta$  (where  $\theta$  is the angle the line makes with the  $x$ -axis) is represented by the matrix  $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$ .



For a transformation matrix  $A$ , the value of the determinant,  $\det(A)$ , is the scale factor of the enlargement of the area from the original shape to the image.

A point is invariant if it does not move under matrix multiplication.

The following transformations are for  $2 \times 2$  matrices.

Transformation	Matrix
Stretch by a scale factor of factor $k$ in the $x$ -direction	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
Stretch by a scale factor of factor $k$ in the $y$ -direction	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
Enlargement with centre of enlargement the origin by a scale factor of factor $k$	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
Reflection in the $x$ -axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in the $y$ -axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Rotation about the origin by $\theta$ in the anticlockwise direction	$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

The following transformations are for  $3 \times 3$  matrices.

Transformation	Matrix
Rotation about the $x$ -axis by angle $\theta$ in the anticlockwise direction	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$
Rotation about the $y$ -axis by angle $\theta$ in the anticlockwise direction	$\begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$
Rotation about the $z$ -axis by angle $\theta$ in the anticlockwise direction	$\begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Enlargement with centre of enlargement the origin by a scale factor of factor $k$	$\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$

**Invariant lines:**

For 2-dimensional cases, use  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ mT \end{pmatrix}$  to determine two equations of the form  $at + bmt = T, ct + dmt = mT$ . Divide to get  $\frac{a+bm}{c+dm} = \frac{1}{m}$ , then solve for value(s) of  $m$  to find the invariant line(s) of the transformation in the form  $y = mx$ .

(3-dimensional case is out of syllabus)