Roots of Polynomials

In this chapter we're going to discuss the relationship of roots of polynomials and its coefficent.

Vieta's theorem

Assume you have a quadratic polynomial

$$ax^2 + bx + c = 0 \ (a \neq 0)$$

Divide by a both sides

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

According to Euler,a polynomial with a degree of x has x solutions,therefore we assume the polynomial's two roots are α, β where $\alpha, \beta \in \mathbb{C}$,then we can rephrase the original polynomial into

$$a(x - \alpha)(x - \beta) = 0$$

divide by a and expand

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

therefore

$$\begin{cases} \alpha + \beta = -\frac{b}{a} \\ \alpha\beta = \frac{c}{a} \end{cases}$$

This is vieta's theorem in quadratic.

Let's do the same thing to cubic polynomials.

$$ax^3 + bx^2 + cx + d = 0$$

We assume the three roots of this polynomial be α, β, γ

Therefore

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$$

$$(x - \alpha)(x - \beta)(x - \gamma) = 0$$

expand

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma=0$$

we get the same thing as shown above

$$\begin{cases}\alpha+\beta+\gamma=-\frac{b}{a}\\ \alpha\beta+\beta\gamma+\gamma\alpha=\frac{a}{c}\\ \alpha\beta\gamma=-\frac{d}{a}\end{cases}$$

Because allat was long, we denote that

$$\alpha+\beta+\gamma+...=\Sigma\alpha$$

(Sum of all roots)

$$\alpha\beta+\beta\gamma+\gamma\alpha+...=\Sigma\alpha\beta$$

(Sum of all possible arrangements of products of 2 roots)

$$\alpha\beta\gamma+...=\Sigma\alpha\beta\gamma$$

(Sum of all possible arrangements of products of 3 roots)

$$\alpha\beta\gamma\delta=\Sigma\alpha\beta\gamma\delta$$

(a polynomial with a degree of n and has m roots inside the Σ notation has ${}^nC_m$ arguments)

(Sum of all possible arrangements of products of 4 roots, 99% of the times only this because we only talk about $\deg(p)\in[2,4]$)

Doing the same thing for quartics (too long and coursebook already got ts) and we get

for

$$ax^4+bx^3+cx^2+dx+e=0$$

$$\begin{cases}\Sigma\alpha=-\frac{b}{a}\\ \Sigma\alpha\beta=\frac{c}{a}\\ \Sigma\alpha\beta\gamma=-\frac{d}{a}\\ \Sigma\alpha\beta\gamma\delta=\frac{e}{a}\end{cases}$$

Questions

(Hodder education CAIE FP1 Example 3.9)

The roots of quartic equation $4z^4 + pz^3 + qz^2 - z + 3 = 0$ are $\alpha, -\alpha, \alpha + \lambda, \alpha - \lambda$ where $\alpha, \lambda \in \mathbb{R}$

(i) express p, q in terms of α, λ

(ii) show that $\alpha = -\frac{1}{2}$ and find the values of p, q

(iii) give the roots of quartic equation

Newton's identities

Denote $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$ for quartic ($S_0 = 4$ since $a^0 = 1$)

(Same thing for cubics)

Useful formula:

$$S_1 = \Sigma\alpha, S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

and

$$S_{-1} = \frac{\Sigma\alpha\beta}{\Sigma\alpha\beta\gamma} \text{ for cubics}$$

$$S_{-1} = \frac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta} \text{ for quartics}$$

(easy to get in the exam, no need to remember)

For $ax^4 + bx^3 + cx^2 + dx + e = 0$ it satifies for all of the roots $\alpha, \beta, \gamma, \delta$ that

$$a\beta^4 + b\beta^3 + c\beta^2 + d\beta + e = 0$$
$$\cdots$$

Adding these 4 equation up we get

$$aS_4 + bS_3 + cS_2 + dS_1 + eS_0 = 0$$

More generally,by multipling x^n to both sides we have

$$aS_{n+4} + bS_{n+3} + cS_{n+2} + dS_{n+1} + eS_n = 0$$

by letting $n = -1$ to solve S_3 and then $n = 0$ to solve S_4

Questions

(2012/O/N/11 Q:11) Roots of equation $x^4 - 3x^2 + 5x - 2 = 0$ are $\alpha, \beta, \gamma, \delta$ and $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$

(i) show that $S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0$

(ii) find the values of S_2, S_4, S_5, S_3

(iii) Hence find the value of

$$\alpha^2(\beta^3 + \gamma^3 + \delta^3) + \beta^2(\gamma^3 + \delta^3 + \alpha^3) + \gamma^2(\delta^3 + \alpha^3 + \beta^3) + \delta^2(\alpha^3 + \beta^3 + \gamma^3)$$

Substitution Method

The roots of cubic equation $2z^3 + 5z^2 - 3z - 2 = 0$ has roots α, β, γ

find the cubic equation with roots $f(\alpha), f(\beta), f(\gamma)$

General solution: $w = f(z), z = f^{-1}(w)$ then substitute $f^{-1}(w)$ back.

Questions

1. The equation $az^3 + bz^2 - cz - d = 0$ has roots α, β, γ, find a cubic equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$ and another cubic equation with roots $\alpha^2, \beta^2, \gamma^2$
2. The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ

   (i) Use the relation $x^2 = -7y$ to show that the equation

   $$49y^3 + 14y^2 - 27y + 7 = 0$$

   has roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$

   (ii) Hence show that $\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2\alpha^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \frac{58}{49}$

   (iii) Find the exact value of $\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\gamma^3\alpha^3} + \frac{\gamma^3}{\alpha^3\beta^3}$