Summation of Series

Tbh this chapter itself isn't hard in the exam way, but if you wanna make hard questions from it with proof by induction, it is extremely easy to make stuff that you have never met before.

What is a series?

Series = Sum of sequence

sequence: $u_n = f(n)$

Series $\sum_{i=1}^{n} f(i)$

Kinds of sequence

let u_n be nth element of a sequence.

if $u_{n+1} > u_n$ then it's an increasing sequence.

if $u_{n+1} < u_n$ then it's an decreasing sequence.

if $\lim_{n\to\infty} u_n = c$ where c is a constant,then it's an convergent sequence.

Ways to determine

Out of syllabus but commonly used in AP and uni-level mathematics.

We only talk about $u_n = f(n)$ such that $f(n)$ is monotonic decreasing and always positive in the interval $[M, +\infty)$ where $M \in \mathbb{Z}^+$

1. Ratio method: if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$ then it is convergent.
2. $\sum_{n=M}^{\infty} f(n)$ converges or diverges with
3. comparison: if $a_n < b_n$. If b_n converges then a_n converges, if a_n diverges then b_n diverges.

Inside the syllabus

Use the limits you've been taught in chapter two ($\frac{1}{n} \to 0$ when $n \to \infty$)

Ew, this kinda suck but like u gotta bear with it because in the exam u gotta write like this to get ur marks.

Some series (not in syllabus)

P series / Harmonic Series

P series: Series that looks like

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

when $p \leq 1$ the series diverge

when $p > 1$ the series converge

especially, when $p = 1$, we call the series harmonic series.

Geometric Series

Series that looks like

$$\sum_{n=1}^{\infty} aq^n$$

(This should ring you a bell about 9709 P1 contents)

When $|q| \geq 1$ the series diverge

When $|q| < 1$ the series converge and the series equals to $\frac{a}{1-r}$(derived from P1)

Alternating Series

Series that look like

$$\sum_{n=0}^{\infty} (-1)^n f(n)$$

where $u_n = f(n)$ and for all n $f(n) > 0$

Formulas

$$\sum_{r=1}^{n} (ar^3 + br^2 + cr + d) = a\sum_{r=1}^{n} r^3 + b\sum_{r=1}^{n} r^2 + c\sum_{r=1}^{n} r + d\sum_{r=1}^{n} 1$$

$$\sum_{r=1}^{n} r^0 = n$$

$$\sum_{r=1}^{n} p = pn$$

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$$

$\sum_{r=1}^{n} r^p$ where ($p \in \mathbb{Z}^+$) can be obtained by using binomial expansion and the results above.

Telescoping sums and method of differences

Telescoping sums are finite sums in which pairs of consecutive terms partly cancel each other, leaving only parts of the initial and final terms.

$$\sum_{n=1}^{N} a_n - a_{n-1} = a_n - a_{n-1} + a_{n-1} - a_{n-2} + \ldots = a_n - a_1$$

Now there's a little problem left for exercise, how do you solve $\sum_{r=1}^{n} r^p$ with telescoping series $n^p = \sum_{r=1}^{n} [r^p - (r-1)^p]$?

Method of differences

It's actually just a pro max of telescoping sums.

e.g: Find

$$\sum_{r=1}^{n} \left(\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2} \right)$$

solution:

$$\sum_{r=1}^{n} \left(\frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2} \right) =$$

$$2 - \frac{3}{2} + \frac{1}{3}$$

$$+\tfrac{2}{2} - \tfrac{3}{3} + \tfrac{1}{4}$$

$$+\tfrac{2}{3} - \tfrac{3}{4} + \tfrac{1}{5}$$

$$+\ldots$$

$$+\frac{2}{n-1} - \frac{3}{n-1} + \frac{1}{n}$$

$$+\tfrac{2}{n-1} \;\; - \tfrac{3}{n} + \tfrac{1}{n+1}$$

$$+\tfrac{2}{n} - \tfrac{3}{n+1} + \tfrac{1}{n+2}$$

notice that $\frac{1}{3} - \frac{3}{3} + \frac{2}{3} = 0$ and $\frac{1}{n} - \frac{3}{n} + \frac{2}{n}$ = 0, therefore everything in between cancels out, the result gives $\frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}$